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A TWO-DIMENSIONAL MODEL OF THE DYNAMICS OF SHARP BENDING OF A NON-LINEARLY ELASTIC PLATE†

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To describe the dynamics of the bending of a thin non-linearly elastic plate, a version of perturbation theory is proposed which correctly takes into account the non-linearity of the medium, the non-uniformity of the deformations along the plate thickness and the boundary conditions on its surface. An effective (2 + 1)-dimensional model is constructed which generalizes the static non-linearly geometrical Föppl–Karman equations. Two-dimensional solitons of the longitudinal deformation are obtained. The conditions for their existence and stability are investigated. © 2003 Elsevier Science Ltd. All rights reserved.

At the present time, it is mainly longitudinal non-linearly elastic waves in an unbounded medium that have been most completely described theoretically (see [1] and the literature cited there). The features of the formation and physical properties of non-linearly elastic excitations and structures in bounded samples and multilayered materials in regions where they undergo sharp transverse bending have hardly been investigated. The derivation of simplified models for non-linearly elastic solids is based on physically obvious geometrical hypotheses, the correctness of which is difficult to assess quantitatively. These approximations are often not satisfied by the boundary conditions on the surfaces of the samples. This has led to an inaccurate transformation with small terms and different ways of writing the fundamental formulae in the non-linear theory of elastic rods, plated [2–5] and shells.

Below we propose a version of perturbation theory for constructing a simplified non-linear (2 + 1)-dimensional model for thin plates, the amplitude of the bending of which is comparable with their thickness. The bending of the plates is assumed to be sharp. A non-linear theory of elasticity is used [6], in which the elastic energy of the medium contains no gradients of the Lagrange strain tensor. Hence, the initial (3 + 1)-dimensional equations of the non-linear theory of elasticity contain no dispersion terms. It is interesting that in the effective (2 + 1)-dimensional model of thin plates, linear and non-linear dispersion terms appear as a result of eliminating the spatial variable, characterizing the non-uniformity of the deformation along the normal to the plate, and as a result of taking into account the boundary conditions on its developed surface. When non-linearity and dispersion effects are balanced it is possible for soliton-like states to form on the surface of the plate. Hence, when constructing simplified (2 + 1)-dimensional non-linear equations for a plate, the boundary conditions on its developed surface must be carefully satisfied. For this purpose we solve a sequence of boundary-value problems in the direction of the normal to the plane of the plate and monitor the accuracy of the parameters characterizing the space-time deformation of the plate, and the geometrical and physical non-linearity of the medium. The boundary conditions on the side faces of the plate are transformed into effective boundary conditions for the (2 + 1)-dimensional model.

The first orders of the proposed perturbation theory in the quasi-static limit lead to well-known equations of the statics of flexible plates [7, 8]. However, such a "geometrical" approximation does not completely describe the non-linear dynamics of sharp bendings of a plate, since, for quasi-one-dimensional deformations, the non-linear two-dimensional equations reduce to linear equations. In order to take into account correctly the effects of the geometrical and physical non-linearity of the medium, we consider subsequent orders of perturbation theory. As a result an effective (2 + 1)-dimensional model is obtained which adequately describes the interaction between the longitudinal deformations, the transverse twistings and bendings of the plate, and also the local changes in the inertial properties of the plate due to its twisting. The transverse loading of the plate is modelled by sources in the simplified equations.

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Below we consider the model problem of the stability and self-resonant vibrations of a plate, which illustrates the fact that the non-linearly geometrical Föppl–Karman approximation [7, 8] is insufficient. Approximations based on geometrical hypotheses or expansions of the displacements in Taylor series in the coordinate perpendicular to the plate surface often do not satisfy the boundary conditions. Hence, they may lead to incorrect assessments and qualitative conclusions in the theoretical description of non-linear, in particular, soliton-like states in the plate, which is illustrated using the example of multisolitons of longitudinal deformation in the plate.

The formation of purely longitudinal deformations of the plate corresponds to more rapid processes and requires a special consideration.

1. FUNDAMENTAL RELATIONS OF THE NON-LINEAR THEORY OF ELASTICITY

In the non-linear theory of finite deformations, the elastic energy of the medium is written in the form of an expansion in invariants of the Lagrange strain tensor with components

$$\eta_{ik} = \frac{1}{2}[\partial_i u_k + \partial_k u_i + \partial_k u_l \partial_i u_l] = \frac{1}{2}\left[\frac{\partial X_n}{\partial x_i}\frac{\partial X_n}{\partial x_k} - \delta_{ik}\right] \tag{1.1}$$

Here x_k are the coordinates of a point mass of the medium before deformation, $X_k = x_k + u_k(\mathbf{x}, t)$ are the coordinates at the same point after deformation ($i, k = 1, 2, 3$), and $\mathbf{u}(\mathbf{x}, t)$ is the displacement vector. For an isotropic medium we choose [7] as the independent invariants of the tensor $\|\eta_{ik}\|$, in terms of which the remaining invariance can be expressed, the following

$$I_1 = \eta_{ll}, \quad I_2 = \eta_{ik}^2, \quad I_3 = \eta_{ik}\eta_{kl}\eta_{li} \tag{1.2}$$

We will represent the expression for the elastic energy of an isotropic non-linear solid in the form

$$W = \int\limits_{V_0} \phi d\mathbf{x}', \quad \phi = \frac{\lambda}{2}I_1^2 + \mu I_2 + \frac{A}{3}I_3 + BI_1I_2 + \frac{C}{3}I_1^3 \tag{1.3}$$

Here ϕ is the energy per unit volume of the solid before deformation. The elastic moduli λ, μ, A, B, and C are assumed to be comparable in order of magnitude. Further, for a plate we will distinguish a region of space-time scales and external loads where, because of the smallness of the deformations, we can neglect the other invariants in the expansion of the energy (1.3).

The dynamic equations for a non-linearly elastic solid can be obtained from Hamilton's principle

$$\delta S + \int\limits_{t_0}^{t} \delta A dt' = 0; \quad S = \int\limits_{t_0}^{t} [K - U] dt' \tag{1.4}$$

The kinetic energy of the system has the form

$$K = \int\limits_{V_0} \frac{\rho_0}{2}(\partial_t u_i)^2 d\mathbf{x}' \tag{1.5}$$

where ρ_0 is the density of the material in the undeformed state (henceforth we will assume $\rho_0 = \text{const}$), and the integration is carried out over the volume V_0 of the undeformed body.

The potential energy U includes the elastic energy W of the body and the energy of its interaction W_1 with external mass forces

$$U = W + W_1; \quad W_1 = -\int\limits_{V_0} \rho_0 P_i u_i d\mathbf{x}' \tag{1.6}$$

Here P_1 is the external mass force and $\rho_0 P_1$ is the force acting on unit volume of the body before deformation.

The work done by the external surface forces has the form [6]

$$\delta A = \int_{\sigma} \delta u_i T_{ij}^{\text{ext}} \det\left\|\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right\| \frac{\partial x^s}{\partial X_j} d\sigma_s \tag{1.7}$$

The integration is carried out over the surface σ of the undeformed body.

Hamilton's principle (1.4) gives the necessary dynamic equations [6]

$$-\rho_0 \partial_t^2 u_i + \partial_s P_{is} + \rho_0 P_i = 0; \quad P_{ij} = \frac{\partial \phi}{\partial \eta_{ij}} + \partial_k u_i \frac{\partial \phi}{\partial \eta_{kj}} \tag{1.8}$$

where $\|P_{ij}\|$ is the Piola–Kirchhoff tensor, and also the boundary conditions, referred to the surface of the undeformed plate

$$P_{is} n_s|_{\sigma} = T_{ij}^{\text{ext}} \det\left\|\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right\| \frac{\partial x_s}{\partial X_j} n_s \Big|_{\sigma} \tag{1.9}$$

Here $\mathbf{n}$ is the vector of the unit normal to the surface σ.

2. CONSTRUCTION OF THE SIMPLIFIED (2 + 1)-DIMENSIONAL EQUATIONS FOR A NON-LINEARLY ELASTIC PLATE

Consider a non-linearly elastic plate, parallel to the x_1Ox_2 plane. Suppose d is the plate thickness along the x_3 axis, l is the characteristic spatial scale of its deformation in the x_1Ox_2 plane, and a and $t_{\text{ch}} = l/\sqrt{\mu/\rho_0}$ are the characteristic amplitude of the displacements and the characteristic deformation time.

We will introduce two small parameters $\epsilon_1 = a/l$ and $\epsilon_2 = d/l$, which reflect the order of smallness of the amplitudes of the displacements and of the plate thickness. In the initial dynamic equations (1.8) we will change to the dimensionless variables

$$\xi_\alpha = x_\alpha/l, \quad \eta = x_3/d, \quad \tau = t/t_{\text{ch}}, \quad u_i = a\bar{u}_i \tag{2.1}$$

They then take the form

$$\mu\epsilon_1\epsilon_2\partial_\tau^2\bar{u}_\alpha = \rho_0 d P_\alpha + \epsilon_2\partial_\beta P_{\alpha\beta} + \partial_\eta P_{\alpha 3} \tag{2.2}$$

$$\mu\epsilon_1\epsilon_2\partial_\tau^2\bar{u}_3 = \rho_0 d P_3 + \epsilon_2\partial_\beta P_{3\beta} + \partial_\eta P_{33} \tag{2.3}$$

Here and henceforth $\alpha, \beta = 1, 2$; $\partial_\alpha = \partial/\partial\xi_\alpha$.

Consider the region of sharp bends of the plate, where the approximate estimate $\epsilon_1 \sim \epsilon_2$ (or $a \sim d$) holds. We will assume that the mass force and external stress on the developed planes ($\eta = \pm{}^1/_2$) of the plate are characterized in order of magnitude by relations

$$d\rho_0 P_3/\mu = O(\epsilon_1^4), \quad d\rho_0 P_\alpha/\mu = O(\epsilon_1^5); \quad T_{33}^{\text{ext}}/\mu = O(\epsilon_1^4), \quad T_{\alpha 3}^{\text{ext}}/\mu = O(\epsilon_1^5) \tag{2.4}$$

The external loading on the side faces of the plate are much greater

$$T_{\alpha\beta}^{\text{ext}}/\mu = O(\epsilon_1^2) \tag{2.5}$$

In this paper the fields $\bar{u}_\alpha$ describe not only local deformations of the material with characteristic scale l, but also the quasi-uniform plane stressed state of the plate, for which $\partial_\beta\bar{u}_\alpha = O(\epsilon_1^2)$. These conditions distinguish the region of physical parameters of the problem in which non-linear dynamics of the plate will be described within the framework of the simpler non-linear (2 + 1)-dimensional model.

To construct the simplified equations we will seek solutions of the initial (3 + 1)-dimensional equations (2.2) and (2.3) in the form

$$\bar{u}_3 = \bar{u}_3^{(0)} + \sum_{n=1}^{\infty} \bar{u}_3^{(n)}, \quad \bar{u}_\alpha = \sum_{n=1}^{\infty} \bar{u}_\alpha^{(n)} \tag{2.6}$$

The superscripts indicate the general order the corresponding terms in the parameters ϵ_1 and ϵ_2 ($\epsilon_1 \sim \epsilon_2$).

We will confine ourselves to considering comparatively slow processes

$$\partial_\tau^2 \bar{u}_3/\bar{u}_3 = T^{\text{ext}}_{\alpha\beta}/\mu = O(\epsilon_1^2) \tag{2.7}$$

In order to indicate the first order of a derivative with respect to time, we will formally make the replacement $\partial_\tau \to \partial_{\tau_1}$.

The following representation corresponds to expansion (2.6)

$$P_{ij} = \sum_{n=1}^{\infty} P_{ij}^{(n)} \tag{2.8}$$

Substituting (2.8) in (2.2) and (2.3) and equating terms of the same order in the parameters ϵ_1 and ϵ_2 to zero, we obtain a chain of equations. The necessary boundary conditions are found by expanding the right-hand side of Eq. (1.9) in the parameters ϵ_1 and ϵ_2. The first orders of perturbation theory give the simple boundary-value problems

$$\partial_\eta P_{33}^{(i)} = 0, \quad P_{33}^{(i)}\big|_{\eta = \pm 1/2} = 0, \quad i = 0, 1, 2, 3 \tag{2.9}$$

$$\partial_\eta P_{\alpha 3}^{(k)} = 0, \quad P_{\alpha 3}^{(k)}\big|_{\eta = \pm 1/2} = 0, \quad k = 1, 2 \tag{2.10}$$

As a result of solving these we obtain expressions for the fields $\bar{u}_k^{(i)}$. The functions $\bar{u}_3^{(i)} = (i = 0, 1)$ do not depend on η. This simplifies the further calculations. Henceforth we will denote functions that are independent of η by a tilde: $\bar{u}_3^{(i)} = \tilde{u}_3^{(i)}$ $(i = 0, 1)$. For $\bar{u}_\alpha^{(k)}$ $(k = 1, 2)$, $\bar{u}_3^{(n)}$ $(n = 2, 3)$ we obtain the expressions

$$\bar{u}_\alpha^{(k)} = -\epsilon_2 \partial_\alpha \tilde{u}_3^{(k-1)} \eta + \tilde{u}_\alpha^{(k)}, \quad k = 1, 2 \tag{2.11}$$

$$\begin{aligned} \bar{u}_3^{(n)} = &-\frac{1}{\lambda + 2\mu}\Big(-\frac{\lambda}{2}(\epsilon_2 \eta)^2 \Delta \tilde{u}_3^{(n-2)} + \\ &+ [\lambda \epsilon_2 \partial_\alpha \tilde{u}_\alpha^{(n-1)} + \epsilon_1 \epsilon_2 (\lambda + \mu)(\partial_\alpha \tilde{u}_3^{(\cdot)} \partial_\alpha \tilde{u}_3^{(\cdot)})^{(n-2)}]\eta\Big) + \tilde{u}_3^{(n)}, \quad n = 2, 3 \end{aligned} \tag{2.12}$$

The functions $\tilde{u}_\alpha^{(k)}$, $\tilde{u}_3^{(n)}$ arose as a result of the integration and are as yet arbitrary. They will be determined by the next orders of perturbation theory.

We can express the components of the two-dimensional strain tensor in terms of the functions $\tilde{u}_\alpha^{(m-1)}$ and $\tilde{u}_3^{(m-2)}$ $(m = 2, 3)$

$$\eta_{\alpha\beta}^{(m)} = \epsilon_{\alpha\beta}^{(m)} - \epsilon_1 \epsilon_2 \eta \partial_\alpha \partial_\beta \tilde{u}_3^{(m-2)}; \quad m = 2, 3 \tag{2.13}$$

The tensor with components

$$\epsilon_{\alpha\beta}^{(m)} = (\epsilon_1/2)[\partial_\alpha \tilde{u}_\beta^{(m-1)} + \partial_\beta \tilde{u}_\alpha^{(m-1)} + \epsilon_1 (\partial_\alpha \tilde{u}_3^{(\cdot)} \partial_\beta \tilde{u}_3^{(\cdot)})^{(m-2)}]$$

describes "plane" deformation that is uniform over the plate thickness. Here and henceforth notation of the form $(\partial_\alpha \tilde{u}_3^{(\cdot)} \partial_\beta \tilde{u}_3^{(\cdot)})^{(m-2)}$ implies the sum of all products of the quantities $\partial_\alpha \tilde{u}_3^{(i)}$ and $\partial_\beta \tilde{u}_3^{(k)}$, which satisfy the limit $i + k = m - 2$. For further calculations the following relation between $\eta_{33}^{(n)}$ and $\eta_{\alpha\alpha}^{(n)}$ will be useful

$$\eta_{33}^{(n)} = -\frac{\lambda}{\lambda + 2\mu}\eta_{\alpha\alpha}^{(n)}, \quad n = 2, 3 \tag{2.14}$$

We will illustrate the general scheme for integrating the equations of perturbation theory using the example of the following boundary-value problem (everywhere henceforth, unless otherwise stated, $m = 3, 4$)

$$\partial_\eta P_{\alpha 3}^{(m)} + \epsilon_2 \partial_\beta P_{\alpha\beta}^{(m-1)} = 0 \tag{2.15}$$

$$P_{\alpha 3}^{(m)}\Big|_{\eta = \pm 1/2} = 0 \tag{2.16}$$

In Eqs (2.15) the tensor $\|P_{\alpha\beta}^{(k)}\|$ ($k = 2, 3$) is symmetrical and can be expressed in terms of functions introduced

$$P_{\alpha\beta}^{(k)} = (\partial\phi/\partial\eta_{\alpha\beta})^{(k)} = \lambda(\eta_{\gamma\gamma}^{(k)} + \eta_{33}^{(k)})\delta_{\alpha\beta} + 2\mu\eta_{\alpha\beta}^{(k)} = -\epsilon_1\epsilon_2\eta\hat{L}_{\alpha\beta}\tilde{u}_3^{(k-2)} + \sigma_{\alpha\beta}^{(k)} \tag{2.17}$$

Here $\hat{L}_{\alpha\beta} = \lambda'\Delta\delta_{\alpha\beta} + 2\mu\partial_\alpha\partial_\beta$ is the differential operator, $\sigma_{\alpha\beta}^{(k)}$ are the components of the symmetrical tensor, characterizing the plane stressed state of the plate

$$\sigma_{\alpha\beta}^{(k)} = \lambda'\epsilon_{\gamma\gamma}^{(k)}\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta}^{(k)} \tag{2.18}$$

and $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$ is the effective modulus of elasticity of plane deformation. The stresses $\sigma_{\alpha\beta}^{(k)}$ give rise to a load on the side faces of the plate. In particular, when there is no load of the order of ϵ_1^3 on the side faces of the plate, we can put $\tilde{u}_3^{(1)} = \tilde{u}_\alpha^{(2)} = \sigma_{\alpha\beta}^{(3)} = 0$, and the perturbation theory is simplified considerably.

We integrate Eqs (2.15) with respect to η in the limits from $\eta = 0$ to a certain value η ($|\eta| \leqslant {}^1/_2$). We obtain

$$P_{\alpha 3}^{(m)}(\eta) - P_{\alpha 3}^{(m)}(0) - \frac{\lambda' + 2\mu}{2}\epsilon(\epsilon_2\eta)^2\Delta\partial_\alpha\tilde{u}_3^{(m-3)} + \epsilon_2\eta\partial_\beta\sigma_{\alpha\beta}^{(m-1)} = 0 \tag{2.19}$$

In relations (2.19) only the dependence on the variable η is indicated explicitly. Assuming $\eta = \pm{}^1/_2$ in (2.19) and taking boundary conditions (2.16) into account, we obtain a system from which we can find $P_{\alpha 3}^{(m)}(0)$ and equations connecting the functions $\tilde{u}_\alpha^{(m-2)}$ and $\tilde{u}_3^{(m-3)}$

$$P_{\alpha 3}^{(m)}(0) = -\frac{\lambda' + 2\mu}{8}\epsilon_1\epsilon_2^2\Delta\partial_\alpha\tilde{u}_3^{(m-3)}, \quad \partial_\beta\sigma_{\alpha\beta}^{(m-1)} = 0 \tag{2.20}$$

Reverting from Eqs (2.20) to Eqs (2.19), we obtain

$$P_{\alpha 3}^{(m)}(\eta) = \frac{\lambda' + 2\mu}{2}\epsilon_1\epsilon_2^2\left(\eta^2 - \frac{1}{4}\right)\Delta\partial_\alpha\tilde{u}_3^{(m-3)} \tag{2.21}$$

On the other hand, by definition (the second relation of (1.8)), we have

$$\begin{aligned} P_{\alpha 3}^{(m)} &= \left(\frac{\partial\phi}{\partial\eta_{\alpha 3}}\right)^{(m)} = 2\mu\eta_{\alpha 3}^{(m)} = \mu\Big(\epsilon_1\partial_\alpha\bar{u}_3^{(m-1)} + \frac{a}{d}\partial_\eta\bar{u}_\alpha^{(m)} + \\ &+ \frac{a}{d}\epsilon_1[(\partial_\alpha\bar{u}_\gamma^{(\cdot)}\partial_\eta\bar{u}_\gamma^{(\cdot)})^{(m-1)} + (\partial_\alpha\bar{u}_3^{(\cdot)}\partial_\eta\bar{u}_3^{(\cdot)})^{(m-1)}]\Big) \end{aligned} \tag{2.22}$$

Note that on the right-hand side of this equation all terms apart from $(a/d)\partial_\eta\bar{u}_\alpha^{(m)}$ are already known. Hence, by combining relations (2.21) and (2.22) we can calculate the longitudinal displacements

$$\begin{aligned} \frac{a}{d}\bar{u}_\alpha^{(m)} &= \epsilon_2^2\epsilon_1\left[\left(1 + \frac{\lambda'}{4\mu}\right)\frac{\eta^3}{3} - \frac{\lambda' + 2\mu}{8}\eta\right]\Delta\partial_\alpha\tilde{u}_3^{(m-3)} + \\ &+ \frac{\epsilon_2\eta^2\lambda'}{4\mu}[\partial_\alpha\epsilon_{\gamma\gamma}^{(m-1)} - \epsilon_1^2\partial_\alpha\tilde{u}_3^{(m-3)}\Delta\tilde{u}_3^{(0)}] + \epsilon_1\eta\Big[-\partial_\alpha\tilde{u}_3^{(m-1)} + \epsilon_1(\partial_\alpha\tilde{u}_\gamma^{(\cdot)}\partial_\gamma\tilde{u}_3^{(\cdot)})^{(m-2)} + \\ &+ \frac{\lambda'}{2\mu}\partial_\alpha\tilde{u}_3^{(m-3)}\epsilon_{\gamma\gamma}^{(2)} + \frac{\epsilon_1^2}{2}\partial_\alpha\tilde{u}_3^{(m-3)}(\partial_\gamma\tilde{u}_3^{(0)})^2\Big] + \frac{a}{d}\tilde{u}_\alpha^{(m)} \end{aligned} \tag{2.23}$$

Here $\tilde{u}_\alpha^{(m)} = \tilde{u}_\alpha^{(m)}(\xi_1, \xi_2, \tau)$ are functions which arise when Eqs (2.21) and (2.22) are integrated.

When $s = 3, 4$ the components $P_{3\beta}^{(s)}$ and $P_{\beta 3}^{(s)}$ are not equal to one another. However, by definition (the second relation of (1.8)), they are related to one another and the function $P_{3\beta}^{(s)}$ is also known. Hence, from the equations of perturbation theory

$$d\rho_0 P_3^{(s+1)} + \epsilon_2 \partial_\beta P_{3\beta}^{(s)} + \partial_\eta P_{33}^{(s+1)} = \mu\epsilon_1\epsilon_2 \partial_{\tau_1}^2 \tilde{u}_3^{(s-3)} \tag{2.24}$$

we can obtain $P_{33}^{(s+1)}$ when $s = 3, 4$. When Eqs (2.24) are integrated with respect to η one must take into account the non-zero boundary conditions on the plate surface (see (1.9) and (2.24))

$$P_{33}^{(s+1)}\Big|_{\eta = \pm 1/2} = [T_{33}^{\text{ext}}]^{(s+1)} \tag{2.25}$$

and the mass forces, if there are any.

The scheme for integrating Eq. (2.24) is no different form that considered in the example of Eq. (2.15). As a result of simple calculations we find $P_{33}^{(n)}(\eta)$ and the evolution equation for the transverse displacements of the plate

$$\begin{aligned} \mu\epsilon_1\epsilon_2 \partial_{\tau_1}^2 \tilde{u}_3^{(n-4)} &= \rho_0 d\langle P_3^{(n)}\rangle + \{P_{33}^{(n)}\} - \\ &- \frac{\lambda' + 2\mu}{12}\epsilon_1\epsilon_2^3 \Delta^2 \tilde{u}_3^{(n-4)} + \epsilon_1\epsilon_2 \partial_\beta [\partial_\gamma \tilde{u}_3^{(\cdot)} \sigma_{\gamma\beta}^{(\cdot)}]^{(n-2)}, \quad n = 4, 5 \end{aligned} \tag{2.26}$$

Here and henceforth, for brevity, we have introduced the notation

$$\langle f\rangle = \int_{-1/2}^{1/2} f(\eta)d\eta, \quad \{f\} = f\left(\eta = \frac{1}{2}\right) - f\left(\eta = -\frac{1}{2}\right) \tag{2.27}$$

In the static case ($\partial_{\tau_1}^2 \tilde{u}_3^{(0)} = 0$) when there is no load of the order of ϵ^3 on the side faces of the plate, the second equation of (2.20) ($m = 3$) and Eq. (2.26) ($n = 4$) form a closed system, identical with the Föppl–Karman equations for a thin plate [7], which are usually derived from the equilibrium conditions when using geometrical hypotheses. This approximation only takes into account the geometrical non-linearity of the medium and is insufficient to investigate the non-linear dynamics of plates. In fact, as a consequence of the second equation of (2.20) ($m = 3$), the last term in Eq. (2.26) ($n = 4$) can be written in the form

$$\epsilon_1\epsilon_2 \partial_\beta [\partial_\alpha \tilde{u}_3^{(0)} \sigma_{\alpha\beta}^{(2)}] = \epsilon_1\epsilon_2 \partial_\alpha \partial_\beta \tilde{u}_3^{(0)} \sigma_{\alpha\beta}^{(2)} + o(\epsilon_1^4)$$

As a result, non-linear equation (2.26) becomes close to linear, and in the case of one-dimensional deformations it reduces to a linear equation. Hence, the non-linear dynamics of thin plates is only completely developed in the next orders of perturbation theory. We emphasize that these relations of dynamics will be due not only to the geometrical non-linearity of the medium but also the physical non-linearity. The physical non-linearity of the medium is characterized by invariance of the third and higher orders in the expansion of the elastic energy (1.3). In order to go outside the framework of the "quasi-linear" approximation (2.20), (2.26), we consider the equations of perturbation theory of the fifth and sixth orders in the parameters ϵ_1 and ϵ_2. These calculations are more tedious but can be carried out using the previous scheme.

The dynamic equations for the longitudinal deformations of the plate have the form

$$\mu\epsilon_1\epsilon_2 \partial_{\tau_1}^2 \tilde{u}_\alpha^{(1)} = F_\alpha^{(5)} + \epsilon_2 \partial_\beta \sigma_{\alpha\beta}^{(4)} + \frac{\epsilon_2^3 \lambda'(\lambda' + 2\mu)}{48\mu} \Delta \partial_\alpha \epsilon_{\gamma\gamma}^{(2)} + \epsilon_2 \partial_\beta \Pi_{\alpha\beta}^{(4)} \tag{2.28}$$

Here

$$\begin{aligned} \sigma_{\alpha\beta}^{(n)} &= \delta_{\alpha\beta}\lambda' \epsilon_{\gamma\gamma}^{(n)} + 2\mu\epsilon_{\alpha\beta}^{(n)} \\ \epsilon_{\alpha\beta}^{(n)} &= \frac{\epsilon_1}{2}(\partial_\alpha \tilde{u}_\beta^{(n-1)} + \partial_\beta \tilde{u}_\alpha^{(n-1)} + \epsilon_1 [\partial_\alpha \tilde{u}_\gamma^{(\cdot)} \partial_\beta \tilde{u}_\gamma^{(\cdot)}]^{(n-2)} + \epsilon_1 [\partial_\alpha \tilde{u}_3^{(\cdot)} \partial_\beta \tilde{u}_3^{(\cdot)}]^{(n-2)}), \quad n = 2, 4 \end{aligned}$$

The effective longitudinal forces

$$F_\alpha^{(5)} = \frac{\epsilon_2\lambda'}{2\mu}\partial_\alpha[d\rho_0\langle\eta P_3^{(4)}\rangle + \{\eta P_{33}^{(4)}\}] + \{P_{\alpha3}^{(5)}\} + d\rho_0\langle P_\alpha^{(5)}\rangle \tag{2.29}$$

are connected with the external loading of the plate, where, by relations (1.9) and (2.4)

$$P_{\alpha3}^{(5)}\big|_{\eta=\pm1/2} = [T_{\alpha3}^{\text{ext}}]^{(5)}$$

It is interesting that the non-uniform transverse loading of the plate by surface and mass forces generates an effective longitudinal force (the first term on the right-hand side of expression (2.29). The components of the tensor $\|\Pi_{\alpha\beta}^{(4)}\|$ have the form

$$\Pi_{\alpha\beta}^{(4)} = [a_1(\epsilon_{\alpha\beta}^{(2)})^2 + a_2(\epsilon_{\gamma\gamma}^{(2)})^2 + b_1(\epsilon_1\epsilon_2\partial_\nu\partial_\mu\tilde{u}_3^{(0)})^2 + b_2(\epsilon_1\epsilon_2\Delta\tilde{u}_3^{(0)})^2]\delta_{\alpha\beta} + \tag{2.30}$$
$$+ 2a_1\epsilon_{\gamma\gamma}^{(2)}\epsilon_{\alpha\beta}^{(2)} + c(\epsilon_1\epsilon_2)^2\Delta\tilde{u}_3^{(0)}\partial_\beta\partial_\alpha\tilde{u}_3^{(0)} + \frac{1}{12}(\lambda' + 2\mu)(\epsilon_1\epsilon_2)^2\partial_\alpha\tilde{u}_3^{(0)}\Delta\partial_\beta\tilde{u}_3^{(0)} + \epsilon_1\partial_\gamma\tilde{u}_\alpha^{(1)}\sigma_{\gamma\beta}^{(2)}$$

where

$$a_1 = \frac{A}{2} + \frac{2\mu B}{\lambda + 2\mu}, \quad a_2 = \frac{1}{(\lambda + 2\mu)^3}[-\lambda^3 A + 6\mu\lambda^2 B + 8\mu^3 C] - \frac{A}{2}$$

$$b_1 = \frac{1}{24}[2a_1 + 3\mu - \lambda'], \quad b_2 = \frac{a_2}{12} - \frac{1}{8}\left[\mu + \frac{\lambda'^2}{2\mu}\right], \quad c = \frac{1}{6}\left[a_1 + \frac{3\mu}{2} + \frac{\lambda'}{4}\right]$$

In relation (2.30) the non-linear terms, which depend on $\epsilon_{\alpha\beta}^{(2)}$, reflect the interaction of the longitudinal deformations of the plate, whereas terms which depend on $\tilde{u}_3^{(0)}$ are induced by transverse bendings of the plate, it twists and curvature.

The dynamic equation for the correction $\tilde{u}_3^{(2)}$ has the form

$$\mu\epsilon_1\epsilon_2\left[\partial_{\tau_1}^2\tilde{u}_3^{(2)} - \frac{\epsilon_2^2}{12}\left(1 - \frac{\lambda'}{2\mu}\right)\partial_{\tau_1}^2\Delta\tilde{u}_3^{(0)}\right] =$$
$$= q^{(6)} - \frac{\epsilon_1\epsilon_2^3}{12}(\lambda' + 2\mu)\epsilon_1\epsilon_2^3\Delta^2\tilde{u}_3^{(2)} + \epsilon_1\epsilon_2\partial_\alpha[\partial_\beta\tilde{u}_3^{(\cdot)}\sigma_{\alpha\beta}^{(\cdot)}]^{(4)} - \tag{2.31}$$
$$- \frac{\epsilon_1\epsilon_2^5}{12}(\lambda' + 2\mu)\left[\frac{1}{5} + \frac{\lambda'}{8\mu}\right]\Delta^3\tilde{u}_3^{(0)} + \epsilon_1\partial_\alpha\Pi_\alpha^{(5)}$$

Here $q^{(6)}$ is the effective two-dimensional field of the external forces, which lead to transverse bendings of the plate

$$q^{(6)} = \epsilon_2\partial_\alpha\left(\{\eta P_{\alpha3}^{(5)}\} + d\rho_0\langle\eta P_\alpha^{(5)}\rangle + \left[1 + \frac{\lambda'}{2\mu}\right]\epsilon_1\partial_\alpha\tilde{u}_3^{(0)}[\{\eta P_{33}^{(4)}\} + d\rho_0\langle\eta p_3^{(4)}\rangle]\right) +$$
$$+ \frac{\epsilon_2^2\lambda'}{4\mu}\Delta(\{\eta^2 P_{33}^{(4)}\} + d\rho_0\langle\eta^2 P_3^{(4)}\rangle) + \{P_{33}^{(6)}\} + d\rho_0\langle P_3^{(6)}\rangle \tag{2.32}$$

The boundary conditions on the plate surface

$$P_{33}^{(6)}\big|_{\eta=\pm1/2} = ([T_{33}^{\text{ext}}]^{(6)} + [T_{33}^{\text{ext}}]^{(4)}\epsilon_1\partial_\alpha\tilde{u}_\alpha^{(1)} + [T_{3\alpha}^{\text{ext}}]^{(5)}\epsilon_1\partial_\alpha\tilde{u}_3^{(0)})\big|_{\eta=\pm1/2} \tag{2.33}$$

follow from expansion (1.9) up to terms of the sixth order in the parameters ϵ_1 and ϵ_2.

The quantity $\Pi_\alpha^{(5)}$ takes into account the effects of non-linear dispersion, and also the interaction of the inhomogeneous twistings, flexure and bendings of the plate with one another and with its longitudinal deformations

$$
\begin{aligned}
\Pi_{\alpha}^{(5)} = & -\epsilon_1\epsilon_2^2\partial_\beta\Big(2\tilde{b}_1\epsilon_{\gamma\gamma}^{(2)}\partial_\beta\partial_\alpha\tilde{u}_3^{(0)} + \tilde{c}\epsilon_{\alpha\beta}^{(2)}\Delta\tilde{u}_3^{(0)} - \frac{\epsilon_1}{12}[\partial_\gamma\tilde{u}_3^{(0)}\hat{L}_{\alpha\beta}\tilde{u}_\gamma^{(1)} + \epsilon_1(\partial_\gamma\tilde{u}_3^{(0)})^2\hat{L}_{\alpha\beta}\tilde{u}_3^{(0)}]\Big) - \\
& -\epsilon_1\epsilon_2^2\partial_\alpha(2\tilde{b}_2\epsilon_{\gamma\gamma}^{(2)}\Delta\tilde{u}_3^{(0)} + \tilde{c}\epsilon_{\gamma\sigma}^{(2)}\partial_\gamma\partial_\sigma\tilde{u}_3^{(0)}) + \\
& +\epsilon_1\partial_\alpha\tilde{u}_3^{(0)}\Big(\Big[\tilde{b}_1(\partial_\gamma\partial_\sigma\tilde{u}_3^{(0)})^2 + \tilde{b}_2(\Delta\tilde{u}_3^{(0)})^2 - \frac{1}{12}(\partial_\sigma\partial_\gamma\tilde{u}_3^{(0)})\hat{L}_{\sigma\gamma}\tilde{u}_3^{(0)}\Big](\epsilon_1\epsilon_2)^2 + a_1(\epsilon_{\gamma\sigma}^{(2)})^2 + \\
& + a_2(\epsilon_{\gamma\gamma}^{(2)})^2\Big) + \epsilon_1\partial_\beta\tilde{u}_3^{(0)}\Big\{\frac{\epsilon_2^2\lambda'}{48\mu}\hat{L}_{\alpha\beta}\epsilon_{\gamma\gamma}^{(2)} + 2a_1\epsilon_{\gamma\gamma}^{(2)}\epsilon_{\alpha\beta}^{(2)} + (\epsilon_1\epsilon_2)^2\tilde{c}\partial_\beta\partial_\alpha\tilde{u}_3^{(0)}\Delta\tilde{u}_3^{(0)}\Big\} - \\
& -\frac{(\epsilon_1\epsilon_2)^2}{12}\partial_\beta\partial_\gamma\tilde{u}_3^{(0)}\hat{L}_{\gamma\beta}\tilde{u}_\alpha^{(1)}
\end{aligned}
\tag{2.34}
$$

where

$$\tilde{b}_1 = b_1 - \frac{\mu}{12}, \quad \tilde{b}_2 = b_2 + \frac{\mu}{12}, \quad \tilde{c} = c - \frac{\lambda' + 2\mu}{12}$$

Note that the second term in the square brackets on the left-hand side of Eq. (2.31) takes into account the change in the inertial properties of the plate due to local changes in its curvature.

The results obtained can be combined and used to construct an effective system of (2 + 1)-dimensional equations for thin plates, the bending of which is comparable with their thickness. These equations define the complete longitudinal and transverse displacements of the plate

$$v_\alpha = \tilde{u}_\alpha^{(1)} + \tilde{u}_\alpha^{(2)} + \tilde{u}_\alpha^{(3)}, \quad v_3 = \tilde{u}_3^{(0)} + \tilde{u}_3^{(1)} + \tilde{u}_3^{(2)}$$

Combining approximations (2.20) and (2.28), (2.26) and (2.31), it is easy to show that, up to terms of the sixth order in the parameters ϵ_1 and ϵ_2 inclusive, the equations of the evolution of the fields v_α and v_3 are

$$\epsilon_1\epsilon_2\mu\partial_{\tau_1}^2 v_\alpha = F_\alpha^{(5)} + \epsilon_2\partial_\beta\sigma_{\alpha\beta} + \frac{\epsilon_2^3\lambda'(\lambda' + 2\mu)}{48\mu}\Delta\partial_\alpha\epsilon_{\gamma\gamma}^{(2)} + \epsilon_2\partial_\beta\Pi_{\alpha\beta}^{(4)} \tag{2.35}$$

$$
\begin{aligned}
& \epsilon_1\epsilon_2\mu\left[\partial_{\tau_1}^2 v_3 - \frac{\epsilon_2^2}{12}\left(1 - \frac{\lambda'}{2\mu}\right)\partial_{\tau_1}^2\Delta v_3\right] = \\
& = q^{\mathrm{eff}} - \frac{\epsilon_1\epsilon_2^3}{12}(\lambda' + 2\mu)\Delta^2 v_3 + \epsilon_1\epsilon_2\partial_\beta[\partial_\alpha v_3\sigma_{\alpha\beta}] - \frac{\epsilon_1\epsilon_2^5}{12}(\lambda' + 2\mu)\left[\frac{1}{5} + \frac{\lambda'}{8\mu}\right]\Delta^3 v_3 + \epsilon_2\partial_\alpha\Pi_\alpha^{(5)}
\end{aligned}
\tag{2.36}
$$

Here we have introduced the two-dimensional deformations and stresses

$$\epsilon_{\alpha\beta} = \frac{\epsilon_1}{2}[\partial_\alpha v_\beta + \partial_\beta v_\alpha + \epsilon_1\partial_\alpha v_\gamma\partial_\beta v_\gamma + \epsilon_1\partial_\alpha v_3\partial_\beta v_3]$$

$$\sigma_{\alpha\beta} = \lambda'\epsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta}$$

The effective two-dimensional forces have the form

$$q^{\mathrm{eff}} = \{P_{33}^{(4)} + P_{33}^{(5)}\} + d\rho_0\langle P_3^{(4)} + P_3^{(5)}\rangle + q^{(6)} \tag{2.37}$$

Expressions for $\epsilon_{\alpha\beta}^{(2)}$, $\Pi_\alpha^{(5)}$ and $\Pi_{\alpha\beta}^{(4)}$ are obtained from the formal replacement $\tilde{u}_\alpha^{(1)} \to v_\alpha$, $\tilde{u}_3^{(0)} \to v_3$ introduced above. The closed system (2.35), (2.36) is an approximation which does not violate the boundary conditions on the plate surface up to terms $O(\epsilon_1^6)$. The effective equations obtained take into account the fundamental non-linear interactions in the plate. They cannot be reduced to linear equations in the case of one-dimensional deformations.

Note that, when relations (2.20) and (2.26) are taken into account the form of system (2.35), (2.36) may be altered. In particular using relations (2.26) one can reduce the overall order of Eq. (2.26) – one can express the derivatives $\Delta^3 v_3$ in terms of second and fourth-order derivatives.

For the model obtained one must formulate effective boundary conditions on the sides of the plate. They can be derived from the variational principle (1.4) after substituting expressions for the fields $\tilde{u}_k^{(n)}$, obtained from perturbation theory, into it. As a result of simple integrations, the variational problem gives the necessary boundary conditions on the sides of the plate, and also the conditions which take into account the concentrated forces at the corners of the plate. It can be shown that such an algorithm, in the fourth order in the parameters ϵ_1 and ϵ_2, leads to boundary conditions that are known for the Föppl–Karman equations [7, 8].

3. A SIMPLE MODEL PROBLEM

Consider a plate loaded by external forces along the x_1 axis

$$T_{11}^{\text{ext}}\big|_{\xi_1=\pm l_1} = O(\epsilon_1^2) + O(\epsilon_1^4)$$

We will assume, for simplicity, that there are no stresses T_{11}^{ext} of order ϵ_1^3 on the side faces of the plate. When $\xi_1 = \pm l_1$ we will assume conditions of hinge support of the sides of the plate. We will also assume that when $\xi_2 = \pm l_2$ the displacements of the plate are bounded by constraints in the x_2 direction, which allow of displacements of the plate in only the x_1 and x_3 directions.

In this case

$$v_1 = v_1(\xi_1), \quad v_3 = v_3(\xi_1), \quad v_2 = 0$$

and the problem is simplified.

Although it is obviously difficult to realise this situation experimentally, nevertheless it is useful to consider it. This example illustrates the insufficiency of the Föppl–Karman approximation [7, 8] and contains features which may be encountered in more complex cases.

Suppose the constant stress T_{11}^{ext} is close to the value T_{11}^{lin}, for which, according to the linear theory, a loss of stability of the plate occurs

$$T_{11}^{\text{lin}}/\mu \sim T_{11}^{\text{ext}}/T_{11}^{\text{lin}} - 1 \sim \epsilon_1^2$$

In the region of neutral stability of the plate the dynamics of the fields v_i will be slow

$$\begin{gathered}\frac{\partial_\tau^2 \bar{v}_i}{\bar{v}_i} \sim \frac{T_{11}^{\text{ext}} - T_{11}^{\text{lin}}}{\mu} \sim \epsilon_1^4 \\ -\frac{\epsilon_2^2}{12}(\lambda' + 2\mu)\partial_1^4 v_3 + \sigma_{11}^{(2)}\partial_1^2 v_3 = o(\epsilon_1^2)\end{gathered} \tag{3.1}$$

In order to stress the second order of the derivative with respect to time we will formally make the replacement $\tau \to \tau_2$.

We will obtain the effective boundary conditions for the plate using the scheme described in the previous section. We will assume that on the sides $\xi_1 = \pm l_1$ of the plate the values of the variations $\delta\tilde{u}_3^{(k)}$, $\delta\partial_1^2\tilde{u}_1^{(k)}$ are fixed, while the variations $\delta\partial_1\tilde{u}_3^{(k)}$, $\delta\tilde{u}_1^{(k+1)}$ are arbitrary ($k = 0, 2$). Under conditions (3.1) the following boundary conditions agree with the equations of perturbation theory

$$\begin{gathered}v_3\big|_{\xi_1=\pm l_1} = \partial_1^2 v_3\big|_{\xi_1=\pm l_1} = 0, \quad \sigma_{11}^{(2)}\big|_{\xi_1=\pm l_1} = [T_{11}^{\text{ext}}]^{(2)} \\ \sigma_{11}^{(4)}\big|_{\xi_1=\pm l_1} = [T_{11}^{\text{ext}}]^{(4)} - ([T_{11}^{\text{ext}}]^{(2)})^2\left(\frac{1}{2\mu} + \frac{3a_1 + a_2}{(\lambda' + 2\mu)^2}\right)\end{gathered} \tag{3.2}$$

Taking relations (3.1) and (3.2) into account, we obtain from the second equation of (2.20) and (2.28)

$$\begin{gathered}\sigma_{11}^{(2)} = [T_{11}^{\text{ext}}]^{(2)} + o(\epsilon_1^2) \\ \sigma_{11}^{(4)} = -(\epsilon_1\epsilon_2)^2\left[b_1 + b_2 + c + \frac{\lambda' + 2\mu}{24}\right](\partial_1^2 v_3)^2 + \sigma_{11}^{(4)}\big|_{\xi_1=\pm l_1} + o(\epsilon_1^4)\end{gathered} \tag{3.3}$$

while from Eq. (2.36) we obtain a closed equation for v_3

$$\partial_{\tau_2}^2 v_3 + a_2\partial_1^2 v_3 + a_4\partial_1^4 v_3 + g\partial_1[\partial_1 v_3(\partial_1^2 v_3)^2] = 0 \qquad (3.4)$$

where

$$a_2 = -\frac{1}{\mu}\left([T_{11}^{\text{ext}}]^{(2)} + [T_{11}^{\text{ext}}]^{(4)} - \frac{1}{2\mu}(([T_{11}^{\text{ext}}]^{(2)})^2)\right)$$

$$a_4 = \frac{\epsilon_2^2}{12\mu}\left(\lambda' + 2\mu + 2[T_{11}^{\text{ext}}]^{(2)}\left(\frac{17}{10} + \frac{3a_1 + a_2}{\lambda' + 2\mu}\right)\right)$$

$$g = \frac{1}{8\mu}(\epsilon_1\epsilon_2)^2(\lambda' + 2\mu)$$

Note that, as a consequence of estimates (3.1), when calculating $\sigma_{11}^{(4)}$ we neglected the inertial terms, and when deriving Eq. (3.4) we omitted the terms $\sim\partial_{\tau_2}^2\partial_1^2 v_3$.

For the problem in question, the non-linearly geometric Föppl–Karman equations can be reduced to linear equations and lead to the threshold stress

$$T_{11}^{\text{lin}} = -\frac{1}{12}(\pi\epsilon_2)^2(\lambda' + 2\mu)$$

for which a neutrally stable solution of the form $v_3 \sim \sin\pi(\xi_1 + l_1)$ appears. In this case the dimensional length of the plate $2L_1$ (the dimensionless length $2l_1$) and the characteristic scale l are connected by the relation $2L_1/l = 2l_1 = n$, where n is a natural number (values of $n > 1$ only arise in the case of an explosive load [9]).

We will seek a solution of the non-linear equation (3.4) in the form

$$v_3 = A(\tau_2)\sin\pi(\xi_1 + l_1) \qquad (3.5)$$

The secular terms in Eq. (3.4) will be eliminated if

$$\partial_{\tau_2}^2 A + \omega_0^2 A - \gamma A^3 = 0; \quad \omega_0^2 = a_4\pi^4 - a_2\pi^2, \quad \gamma = g\pi^6/4 \qquad (3.6)$$

Suppose $A = 1$, $\partial_{\tau_2} A = 0$ when $\tau_2 = 0$ (the amplitude a of the displacements of the plate is introduced in the definition of ϵ_1). Equation (3.6) allows of the first integral

$$2(\partial_{\tau_2} A)^2 = \gamma(A^2 - 1)(A^2 - \beta); \quad \beta + 1 = 2\omega_0^2/\gamma \qquad (3.7)$$

Equation (3.7) has a bounded solution only when $\beta \geqslant 1$ [10]. The neutral stability of the plate corresponds to the equality, Hence we obtain the threshold value of the modulus of the external loading

$$|T_{11}^{\text{nl}}| = |T_{11}^{\text{lin}}|\left(1 - \frac{(\pi\epsilon_2)^2}{6}\left[\frac{17}{10} + \frac{3a_1 + a_2}{\lambda' + 2\mu}\right]\right) - \frac{(T_{11}^{\text{lin}})^2}{2\mu} - \frac{\lambda' + 2\mu}{32}(\pi^2\epsilon_1\epsilon_2)^2$$

Hence, loss of stability of a non-linearly elastic plate occurs for a load which differs from that obtained from the 'quasi-linear" theory [7, 8].

Suppose the longitudinal load on the plate is less than the critical value: $\omega_0^2 > \gamma$. We will discuss the forced vibrations of a plate acted upon by a small resonant load with a developed surface

$$q^{\text{eff}} = T_{33}^{\text{ext}}\Big| = \alpha q(x, \tau_2)\cos\phi(\tau_2)$$

Here we have introduced a new dimensionless parameter α, characterizing the smallness of the external action ($0 < \alpha \leqslant 1$). Suppose the amplitude of the external force varies slowly compared with its frequency: $\partial_{\tau_2} q/\partial_{\tau_2}\phi = o(1)$.

In this case, instead of Eq. (3.6) we have the following for the amplitude of the vibrations of the plate

$$\partial_{\tau_2}^2 A + \omega_0^2 A - \gamma A^3 = \alpha f(\tau_2)\cos\phi(\tau_2) \qquad (3.8)$$

where

$$f(\tau) = \frac{1}{l_1\epsilon_1\epsilon_2\mu}\int_{-l_1}^{l_1} q(\tau, \xi_1)\sin\pi(\xi_1 + l_1)d\xi_1$$

The properties of the solutions of non-linear equation (3.8) with zero initial data $(A, \partial_{\tau_2}A) = (0, 0)$ were investigated in [11] and the conditions for which the energy of the system increases, although the external force remains small, were obtained.

Unlike resonance, in the linear problem in order for non-linear self-resonance to occur it is necessary for the amplitude of the external force to exceed a certain threshold value. Moreover, in the non-linear problem the frequency of natural vibrations of the plate decreases as the amplitude increases. Hence, to achieve self-resonance at the first stage (while the amplitude of the vibrations of the plate is still small) one must slowly vary the phase of the inducing forces

$$\phi(\tau_2) = \omega_0\tau_2 + \alpha^{-2\lambda/3}\Phi(\alpha^{2(1+\lambda)/3}\tau_2), \quad \lambda \geq 0 \tag{3.9}$$

The conditions which the functions f and Φ must satisfy in order for the problem to have increasing solutions as $\tau_2 \to \infty$, were obtained in [11]. These conditions depend on the rate of monotonic change in the frequency $\partial_{\tau_2}\phi$ of the inducing force and correspond to hard ($\lambda = 0$) and soft ($\lambda > 0$) modes of self-resonance.

4. NON-LINEAR DYNAMICS OF LONGITUDINAL DEFORMATIONS OF THE PLATE

Longitudinal deformations, which occur in the plane of the plate and are not accompanied by bending of the plate, are a particular form of the deformations of a thin plate. Unlike transverse bendings of the plate, these are comparatively rapid processes. The equations of perturbation theory given above must therefore be changed.

Suppose the previous estimates hold for the change in the fields u_α in space and for the value of the external loading, while estimate (2.7), characterizing the change in the displacements with time, is replaced by $\partial_\tau\bar{u}_\alpha/\bar{u}_\alpha = O(1)$. In this case the effective equations of the dynamics of longitudinal deformations of the plate are obtained by a small modification of the previous calculation scheme. On the one hand, additional inertial terms appear in the equations of perturbation theory. On the other, all terms not containing derivatives with respect to time are obtained from the previous equations provided $\tilde{u}_3^{(k)} = 0$ $(k = 0, 1, 2)$. We will point out the key factors.

Essentially, for the rapid processes being considered, the inertial properties of the plate and the two-dimensional stresses $\sigma_{\alpha\beta}^{(k+1)}$ $(k = 1, 2)$ turn out to be related:

$$\mu\epsilon_1\partial_\tau^2\tilde{u}_\alpha^{(k)} = \partial_\beta\sigma_{\alpha\beta}^{(k+1)} + o(\epsilon_1^{k+1}) \tag{4.1}$$

The relation between the stresses $\sigma_{\alpha\beta}^{(k+1)}$ and the displacements is the same as relation (2.18) with $\tilde{u}_3^{(k)} = 0$ $(k = 0, 1, 2)$. From the equations of evolution for the fields $\tilde{u}_\alpha^{(3)}$, taking into account the constraint

$$\mu\partial_\tau^2\epsilon_{\gamma\gamma}^{(2)} = (\lambda' + 2\mu)\Delta\epsilon_{\gamma\gamma}^{(2)} + o(\epsilon_1^2)$$

which follows from relation (4.1), we obtain, after integration with respect to η

$$\mu\epsilon_1\epsilon_2\partial_\tau^2\tilde{u}_\alpha^{(3)} = \frac{\epsilon_2^3}{12}\mu\left(\frac{\lambda'}{2\mu}\right)^2\partial_\tau^2\partial_\alpha\epsilon_{\gamma\gamma}^{(2)} + F_\alpha^{(5)} + \epsilon_2\partial_\beta\sigma_{\alpha\beta}^{(4)} + \epsilon_2\partial_\beta\Pi_{\alpha\beta}^{(4)} \tag{4.2}$$

Expressions for $\epsilon_{\alpha\beta}^{(2)}$, $\sigma_{\alpha\beta}^{(4)}$, $\Pi_{\alpha\beta}^{(4)}$, are obtained from the previous ones when $\tilde{u}_3^{(k)} = 0$ $(k = 0, 1, 2)$.

The effective non-linear equations for the resulting displacements $v_\alpha = \tilde{u}_\alpha^{(1)} + \tilde{u}_\alpha^{(2)} + \tilde{u}_\alpha^{(3)}$ are obtained by combining expressions (4.1) and (4.2)

$$\mu\epsilon_1\epsilon_2\partial_\tau^2 v_\alpha = F_\alpha^{(5)} + \epsilon_2\partial_\beta\sigma_{\alpha\beta} + \frac{\epsilon_1^3}{12}\mu\left(\frac{\lambda'}{2\mu}\right)^2\partial_\tau^2\partial_\alpha\epsilon_{\gamma\gamma}^{(2)} + \epsilon_2\partial_\beta\Pi_{\alpha\beta}^{(4)} \tag{4.3}$$

where

$$\sigma_{\alpha\beta} = \lambda'\delta_{\alpha\beta}\epsilon_{\gamma\gamma} + 2\mu\epsilon_{\alpha\beta}$$

$$\epsilon_{\alpha\beta} = \frac{\epsilon_1}{2}[\partial_\alpha v_\beta + \partial_\beta v_\alpha + \epsilon_1 \partial_\alpha v_\gamma \partial_\beta v_\gamma], \quad \epsilon^{(2)}_{\alpha\beta} = \frac{\epsilon_1}{2}[\partial_\alpha v_\beta + \partial_\beta v_\alpha]$$

while in the expression $\Pi^{(4)}_{\alpha\beta}$ we must make the formal replacement $\tilde{u}^{(1)}_\alpha \to v_\alpha$.

Equations similar to (4.3) were derived previously in [12] from the variational principle using the hypothesis of the generalized plane stressed state. This hypothesis corresponds exactly to relation (2.12) when $n = 2$ and $\tilde{u}^{(k)}_3 = 0$ ($k = 0, 2$). However, approximation (2.12), if we neglect the correction $\tilde{u}^{(3)}_\alpha$ by itself violates boundary condition $P^{(3)}_{\alpha 3}$ ($\eta = \pm^1/_2$) $= 0$ on the plate surface. At the same time the inclusion of $\tilde{u}^{(3)}_\alpha$ leads to the disappearance of the linear dispersion term $\sim\partial_\alpha\Delta\epsilon^{(2)}_{\gamma\gamma}$ in the equations of [12]. In order respects the equations of [12] are identical with Eqs (4.3). The parameters β_1 and β_2 from [12] are related to the moduli of elasticity a_1 and a_2 of the present paper by the relations

$$\beta_1 = a_1/(\lambda' + 2\mu), \quad \beta_2 = 2(3a_1 + a_2)/(3[\lambda' + 2\mu])$$

Since the linear dispersion terms are responsible for the formation of soliton-like states, we will discuss the possibility of the formation of longitudinal-deformation solitons in the plate. We will consider the special case when the longitudinal displacements of the plate depend only slightly on the spatial coordinate ξ_2, and the displacements along the x_2 axis are small compared with the displacements along the x_1 axis: $v_2/v_l \sim \partial_2 v_\alpha/\partial_1 v_\alpha \sim \epsilon_1$. Moreover, we will neglect surface and mass forces.

Note, that when describing processes which change more slowly in space than the ones considered, there is no need to revert once again to the initial (3 + 1)-dimensional equations and to reconstruct the perturbation theory. It is easier to carry out the necessary reduction within the framework of the effective equations (4.3).

In this case, in the principal approximation, there are no local rotations of the medium around the x_3 axis, and hence the shear deformations satisfy the condition $\partial_1 v_2 \simeq \partial_2 v_1$. Taking this into account, system (4.3) can be reduced to a (2 + 1)-dimensional equation for the field $\phi = \partial_1 v_1$

$$\partial^2_\tau \phi = \left[\frac{\lambda' + 2\mu}{\mu}\Delta + \frac{\epsilon_2^2}{12}\left(\frac{\lambda'}{2\mu}\right)^2 \partial^2_\tau \partial^2_1\right]\phi + g\partial^2_1\phi^2 \tag{4.4}$$

In the long-wave limit for perturbations, which propagate with velocities close to the velocity of sound $s = \sqrt{(\lambda' + 2\mu)/\mu}$ (in dimensionless variables), Eq. (4.4) can be simplified by using the approximation $\partial^2_\tau\phi \approx s^2\partial^2_1\phi + o(\epsilon_1)$. As a result we have

$$\partial^2_\tau\phi = [\alpha^{(2,0)}\Delta - \alpha^{(4,0)}\partial^4_1]\phi + g\partial^2_1\phi^2 \tag{4.5}$$

where

$$\alpha^{(2,0)} = \frac{\lambda' + 2\mu}{\mu}, \quad \alpha^{(4,0)} = -\frac{\epsilon_2^2}{12}\left(\frac{\lambda' s}{2\mu}\right)^2$$

$$g = \frac{\epsilon_1}{\mu}\left[3a_1 + a_2 + \frac{3}{2}(\lambda' + 2\mu)\right]$$

If, in Eq. (4.5), we neglect the dependence of the field ϕ on the spatial coordinate ξ_2, it reduces to the completely integrable Boussinesq model. If we confine ourselves to considering waves moving in one direction along the x_1 axis, with velocities close to the velocity of sound, model (4.5) can be reduced to the (2 + 1)-dimensional integrable Kadomtsev–Petviashvili model.

In the general case, it has been shown [13] that Eq. (4.5) allows of a Bäcklund transformation. If ϕ_0 is a certain solution of this equation and the function f satisfies the equation

$$[D^2_\tau - \alpha^{(2,0)}(D^2_1 + D^2_2) + \alpha^{(4,0)}D^4_1 - 2g\phi_0 D^2_1]f\cdot f = 0 \tag{4.6}$$

then

$$\phi_1 = \phi_0 - 6\alpha^{(4,0)}g^{-1}\partial_1^2 \ln f$$

will also be a solution of Eq. (4.5). Here $D_\tau^2 f \cdot f = (\partial_\tau - \partial_{\tau'})f(\tau)f(\tau')|_{\tau=\tau'}$, etc. are Hirota operators. The bilinear form enables us to use Hirota's method to obtain (2 + 1)-dimensional soliton-like solutions [13]. In particular, we obtain the N-soliton exponential solution

$$\begin{aligned} \phi &= -6\alpha^{(4,0)}g^{-1}\partial_1^2 \ln f \\ f &= \sum_{\mu=0,1} \exp\left[\sum_{i>j} A_{ij}\mu_i\mu_j + \sum_i \mu_i\eta_i\right] \\ \exp A_{ij} &= -[(\Omega_i-\Omega_j)^2 - k^2(p_i-p_j, q_i-q_j)][(\Omega_i+\Omega_j)^2 - k^2(p_i+p_j, q_i+q_j)]^{-1} \\ \exp\eta_i &= \exp[\Omega_i\tau + p_i\xi_1 + q_i\xi_2 + \eta_{0i}] \\ k^2(p,q) &= \alpha^{(2,0)}[p^2+q^2] - \alpha^{(4,0)}p^4, \quad \Omega_i = k(p_i, q_i) \end{aligned} \tag{4.7}$$

Here $\sum_{\mu=0,1}$ denotes summation over all possible combinations of $\mu = 0, 1$, $\sum_{i>j}$ denotes summation over all possible pairs of N elements, and $\sum_i$ denotes summation over i from $i = 1$ to $i = N$. The parameters $\Omega_i, p_i, q_i, \eta_{0i}$ must satisfy reductions which guarantee that ϕ is real.

When $N = 2M$, and the parameters are related pairwise such that $\Omega_s = \Omega^*_{s+M}$, $P_s = P^*_{s+M}$, etc. ($s = 1, 2, \ldots M$), expression (4.7) describes an ensemble of M two-dimensional "pulsating" solitons. These solitons become singular at certain space-time points, and hence it is difficult to give them a physical interpretation.

For real $p_i, q_i, \Omega_i, \eta_{0i}$ the solution describes elastic paired collisions of N quasi-one-dimensional "plane" solitons of the type

$$\phi = -\frac{6\alpha^{(4,0)}d^2}{g\,\mathrm{ch}^2\Theta}, \quad \Theta = d(\xi_1 + ut), \quad d^2 = \frac{\alpha^{(2,0)} - u^2}{4\alpha^{(4,0)}} > 0$$

where u the soliton velocity. "Plane" solitons are non-singular. Their stability or instability to two-dimensional perturbations depends on the sign of the quantity $\alpha^{(2,0)}\alpha^{(4,0)}$. In this case $\alpha^{(2,0)}\alpha^{(4,0)} < 0$, and "plane" solitons, moving with supersonic velocities ($u^2 > s^2$), are stable to two-dimensional perturbations [13].

It was shown in [13] that model (4.5) also allows of "cigar-shaped" polynomial solitons. In this case these solitons are unstable to two-dimensional perturbations.

The above analysis agrees with the results of experiments on the observation of longitudinal deformation solitons in a plate [14].

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